Designing Flexible Systems Using a New Notion of Submodularity

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We study the problem of optimal flexibility capacity portfolio selection by introducing a new notion of submodularity for correspondences, which extends the classical notion of submodular functions. In particular, we prove that the correspondence that maps flexible resources to the set of demands that they can process is submodular, and use the properties of submodular correspondences to compare different flexibility configurations and derive insights into the optimal capacity portfolio.

Key words: Flexibility, submodularity, correspondences

1. Introduction

We study the problem of capacity selection for a firm that produces n different types of products and has access to resources with different production capabilities. The ability of a resource to produce different product types is termed as its flexibility. Jordan and Graves (1995) studied this problem using a newsvendor framework in which they proposed a chaining configuration consisting of bi-flexible resources (that can produce two product types), and showed that chaining configurations can perform quite well.

In this paper, we study this classical problem with the goal of developing a means of comparing different resource configurations that can be used to draw insights into the structure of optimal resource configurations. To do so, we first extend the notion of submodular functions from real-valued functions to correspondences, and then use properties of these correspondences to investigate optimal capacity portfolios that minimize the system cost while meeting specified constraints. An important feature of our work is that we allow the shortage cost functions and constraints to depend on both the realized and satisfied demand in any manner. Under some reasonable conditions on the costs of investing in resources (including linear economies of scope), we obtain a nice structure on the optimal capacity portfolio: resources invested in must have “similar” flexibility, and resources with lower flexibility must be more productive than those with higher flexibility.

This paper consists of two main sections: Section 2 introduces the notion of submodular correspondences, and Section 3 uses this notion in optimizing flexible capacity portfolios. The focus of this paper is on single-period newsvendor models and we discuss the implications of our results for other models in Section 4.

Notation. We use $\mathcal{P}(A)$ to denote the power set of A. For sets A and B, we use the notation $A+B \equiv \{x+y|x \in A, y \in B\}$. Further, we use the notation $m \times A = A+(m-1) \times A$ for $m \in \mathbb{Z}, m > 1$, $mA \equiv \{mx|x \in A\}$ for $m \in \mathbb{R}_+$ and $Co(A)$ as the convex hull of A. For vectors $x, y \in \mathbb{R}^n$, we use $x \vee y$ to denote the component-wise maximum and $x \wedge y$ to denote the component-wise minimum of x and y. Finally, we say that a set $X \subseteq \mathbb{R}^n$ is a lattice, if for any $x, y \in X$ we have $x \vee y, x \wedge y \in X$.

2. Submodularity for Correspondences

We begin by recalling the definition of submodularity for real-valued functions. Let $X \subseteq \mathbb{R}^n$ be a lattice. Then, a function $f : X \to \mathbb{R}$ is submodular if

$$f(x \vee y) + f(x \wedge y) \leq f(x) + f(y), \text{ for all } x, y \in X.$$

Further, if $-f$ is submodular then f is said to be supermodular, and if the inequality is changed to an equality the function is said to be modular. For a more detailed discussion on submodularity, see Chapter 2 of Simchi-Levi, Chen and Bramel (2005).

We extend this notion of submodularity to correspondences as follows:

Definition 1. A correspondence $\mathcal{F} : X \to \mathcal{P}(\mathbb{R}^n)$ is *submodular* if

$$\mathcal{F}(x \vee y) + \mathcal{F}(x \wedge y) \subseteq \mathcal{F}(x) + \mathcal{F}(y). \tag{1}$$

If the set inclusion is reversed in (1), then the correspondence $\mathcal{F}$ is said to be supermodular, and if both sets are equal then $\mathcal{F}$ is said to be modular. This new notion is a generalization because a function $f : X \to \mathbb{R}$ is submodular if and only if the correspondence $\mathcal{F}(x) = (-\infty, f(x))$ is submodular. The following are some properties of submodular correspondences.

Proposition 1 **(Properties of submodular correspondences)**.

1. *If correspondences $\mathcal{F}_i : X \to \mathcal{P}(\mathbb{R}^n)$ for $i = 1, \ldots, n$ are submodular, then $\mathcal{F} = \sum_{i=1}^n \alpha_i \mathcal{F}_i$ is submodular for $\alpha_i \in \mathbb{R}_+$ for $i = 1, 2, \ldots, n$.*

2. *If a correspondence $\mathcal{F} : X \to \mathcal{P}(\mathbb{R}^n)$ is submodular, then the convex hull of $\mathcal{F}$ is also submodular, i.e., the correspondence $\mathcal{G} : X \to \mathcal{P}(\mathbb{R}^n)$ defined as $\mathcal{G}(x) = Co(\mathcal{F}(x))$ for all $x \in X$ is submodular.*

2.1. Example 1: A submodular correspondence.

Define $X = \{(0,0),(1,0),(0,1),(1,1)\}$ and the correspondence $\mathcal{F}(\cdot)$ on X as follows

$$\mathcal{F}(x) = \begin{cases} \{(0,0)\}, & \text{if } x = (0,0), \\ \{(0,0),(1,0)\}, & \text{if } x = (1,0), \\ \{(0,0),(0,1)\}, & \text{if } x = (0,1), \\ \{(0,0),(1,0),(0,1)\}, & \text{if } x = (1,1). \end{cases}$$

Then, $\mathcal{F}$ is a submodular correspondence. To see this, observe that for all $x, y \in X$ such that $\{x,y\} \neq \{(0,1),(1,0)\}$, we either have $x \vee y = x$ and $x \wedge y = y$ or $x \vee y = y$ and $x \wedge y = x$, which implies that $\mathcal{F}(x \vee y) + \mathcal{F}(x \wedge y) = \mathcal{F}(x) + \mathcal{F}(y)$. Thus, to ascertain the submodularity of $\mathcal{F}$, we only need to check relation (1) for $x = (0,1)$ and $y = (1,0)$. At these values, we have

$$\mathcal{F}(x) + \mathcal{F}(y) = X \supset \{(0,0),(0,1),(1,0)\} = \mathcal{F}(x \vee y) + \mathcal{F}(x \wedge y).$$

Thus, $\mathcal{F}$ is a submodular correspondence.

3. Submodular Correspondences and Optimal Flexibility Configurations

This section explores the implications of submodular correspondences for selecting optimal flexibility portfolios. Section 3.1 relates the concept of submodular correspondences to the performance of flexible resources. Sections 3.2 and 3.3 then formalize this connection in multi-product newsvendor settings and derive properties of the optimal flexibility portfolio. Finally, Section 3.4 generalizes our results to a setting in which resources can also differ in their productivity or processing efficiency.

3.1. A two-product resource allocation problem in the setting of Example 1

The simple submodular correspondence described in Example 1, $\mathcal{F}$, (cf. Section 2.1) can be mapped to a classical operations problem of matching supply and demand in a two product setting with three resources: one dedicated resource that can satisfy product-1 demand, one dedicated resource that can satisfy product-2 demand, and one flexible resource that can satisfy either product-1 or product-2 demand, represented by $(1,0)$, $(0,1)$, and $(1,1)$ respectively. (The element (0,0) represents a null resource that satisfies no demand.) The correspondence $\mathcal{F}(r)$ thus enumerates all possible demand vectors that can be met by a resource r, and in this sense represents its *performance region*. For instance, $\mathcal{F}(1,1) = \{(0,0),(0,1),(1,0)\}$ means that the resource $(1,1)$ can either satisfy zero demand, denoted by $(0,0)$, or product 1 demand $(1,0)$, or product 2 demand $(0,1)$. The performance region of a portfolio of resources can then be obtained by simply adding the performance regions for individual resources in the portfolio. That is, for a portfolio consisting of resources $r_1, r_2, \ldots, r_k$, the performance region of the portfolio equals $\sum_{i=1}^{k} \mathcal{F}(r_i)$.

In this two-product setting, it is obvious that, disregarding resource cost considerations, two dedicated resources should be preferred to the flexible resource alone in the sense that the former can satisfy more demand. This relation is mathematically equivalent to the relation $\mathcal{F}(1,0)+\mathcal{F}(0,1)\supset\mathcal{F}(1,1)=\mathcal{F}(1,1)+\mathcal{F}(0,0)$, which is precisely the condition for the submodularity of the performance region $\mathcal{F}$. Thus, in this simple resource allocation scenario, we find that the underlying performance region is a submodular correspondence. We next prove that this property holds for general, multi-product settings as well.

3.2. The model

Consider a system with n product types with random (potentially correlated) demand. There are 2^n-1 different types of potential resources, each capable of handling different set of products. We represent a resource by the product types it can process, so a resource r, where $r\in\{0,1\}^n$ processes product type i if any only if $r_i=1$. Thus, a resource r is more flexible than resource s if $r\geq s$.[1] Further, the number of different product types that resource r can process is given by $\sum_{i=1}^n r_i$.

To formulate the capacity portfolio optimization problem, we need to attribute costs to resources and penalties or quality of service constraints for unmet demand, and then minimize the overall expected cost. We denote the cost incurred as a function of the realized demand and the demand that is met by $C:\mathbb{R}^{2n}\to\mathbb{R}$. This cost function includes the typical functions that attribute a penalty cost for each unit of demand that is not met. Further, the system manager may face a number of constraints that depend on the realized and satisfied demands, such as fill rates, in-stock rates, etc. We denote the set of these constraints by G so that each element $g\in G:\mathbb{R}^{2n}\to\mathbb{R}$ respresents a constraint. We denote the per unit cost of resource $r\in\{0,1\}^n$ as $c(r)$ so that the total resource cost of a portfolio $K=(K_r\in\mathbb{Z}_+ : r\in\{0,1\}^n)$ equals $\sum_{r\in\{0,1\}^n}c(r)K_r$. The optimization problem is then given by:

$$\min_{K,x,y}\ \mathbb{E}_D\left[C(D,x(D))\right]+\sum_{r\in\{0,1\}^n}c(r)K_r, \tag{2}$$

$$s.t.\quad \mathbb{E}_D\left[g(D,x(D))\right]\geq 0,\ \text{for } g\in G, \tag{3}$$

$$x_i(D)\leq D_i,\ \text{for } i=1,2,\ldots,n, \tag{4}$$

$$x_i(D)\leq \sum_{r\in\{0,1\}^n} y_{i,r}(D),\ \text{for } i=1,2,\ldots,n, \tag{5}$$

$$\sum_{i=1}^n y_{i,r}(D)\leq K_r,\ \text{for } r\in\{0,1\}^n, \tag{6}$$

$$y_{i,r}(D)\leq K_r r_i, \text{for } i=1,2,\ldots,n \text{ and } r\in\{0,1\}^n. \tag{7}$$

[1] Note that is a partial ordering as there may be resources r and s which cannot be compared in this fashion.

For each demand realization $D \in \mathbb{Z}_+^n$, $y_{i,r}(D) \in \mathbb{Z}_+$ refers to the capacity of resource r allocated to product type i and $x_i(D) \in \mathbb{Z}_+$ refers to the total satisfied demand of product type i. (4) places the constraint that $x_i(D)$ must be less than the product demand, (5) constrains the total demand satisfied for each product type to be less than the total capacity available to process this product type, (6) constrains the capacity allocation from each resource to be less than the number of available units of that resource, and finally (7) ensures that resource r cannot be allocated to product i if it cannot process it, that is, if $r_i = 0$. For simplicity, we assume that an optimal capacity portfolio exists.

3.3. Analysis

We will use the performance region approach to characterize the solution to the capacity portfolio optimization problem. To do so, we first denote $\mathcal{F} : \{0,1\}^n \to \mathcal{P}(\{0,1\}^n)$ as the correspondence that maps the resources to the demand vectors that they can satisfy. Then, we can write

$$\mathcal{F}(r) = \{y \in \{0,1\}^n | \sum_{i=1}^{n} y_i \leq 1 \text{ and } y_i \leq r_i \text{ for all } 1 \leq i \leq n\}. \quad (8)$$

We then obtain the following result.

Proposition 2. *The performance region $\mathcal{F}$ is a submodular correspondence.*

The correspondence $\mathcal{F}$ is defined for individual resources and can be combined to obtain the performance region for any portfolio of resources in the following fashion. If the portfolio contains K_r units of resource r, then these can be used to satisfy any demand that lies in the set $K_r \times \mathcal{F}(r)$. Thus, the performance region of a portfolio of resources, K, is $\sum_{r \in \{0,1\}^n} K_r \times \mathcal{F}(r)$.

Using the above result on the submodularity of the performance region, we obtain the following insight into the optimal portfolio.

Proposition 3. *If the resource cost function $c(r)$ is a supermodular function, then there exists an optimal portfolio such that for any pair of resources u, v invested in, if v is more flexible than u, i.e., $v \geq u$, then v can process at most one more product type than u.*

Note that one may assign the null resource a positive cost to ensure that the cost function $c(r)$ is supermodular.

While a formal proof of this result is in the Appendix, the intuition is easy to see with the following example. Consider a pair of resources that can process product types $\{1\}$ and $\{1,2,3\}$, in which the latter is more flexible and can process two product types more than the former. The submodularity of the performance region implies that the total demand processed by these two resources is less than that processed by a pair of resources that can process product types $\{1,2\}$ and $\{1,3\}$. Further, if the resource cost function is supermodular, then the latter pair of resources have a lower total cost as well. Thus, implying that resources $\{1\}$ and $\{1,2,3\}$ cannot simultaneously be part of the optimal capacity portfolio.

Proposition 3 proves that for supermodular resource cost functions (which includes the case of affine functions), the ideal resource configuration should consist of resources that are similar to each other in their abilities. This result holds for any possible demand realization, and thus does not depend on how one incorporates the demand into the objective function. So, this result remains valid when one considers shortage costs of unmet demand or quality of service constraints, and in this sense is quite powerful. Note that the chaining configurations proposed in Jordan and Graves (1995) comprise resources with similar flexibility, and is consistent with this result.

The result for the newsvendor setting discussed in Bassamboo, Randhawa and Van Mieghem (2010) is a special case. To see this, note that we can model the optimization problem in that paper as follows: Set $C(D,x(D)) = \sum_{i=1}^{n} p_i[D_i - x_i(D)]^+$, where p_i is the penalty associated with each unmet demand of product i; $G = \phi$, the empty set, because there are no constraints; and $c(r) = c_1(1 + (\sum_{i=1}^{n} r_i - 1)\delta)$, where c_1 is the cost of resources dedicated to a single product and δ is the flexibility premium for processing each additional product. Noting that $c(r)$ is a modular function, we can apply Proposition 3 to obtain that the optimal flexibility portfolio must invest in resources at adjacent levels of flexibility.

3.4. Extending the model

We extend the model of Section 3.2 to allow resources to have different levels of productivity. In particular, one unit of a resource can now be used to process different units of various product types. This can be perceived as the resource's efficiency in processing the product types, that is, if one unit of a resource can process more products of one type than that of another resource, then it is said to be more productive in processing that product type.

We now represent a resource by the amount of each product that can be processed per unit, so each unit of a resource $r \in \mathbb{R}_+^n$ processes r_i units of product type i (note that if the resource cannot process product i, then $r_i = 0$). We assume that the system manager has access to resources that lie in a finite set $S \subset \mathbb{R}_+^n$. The optimization problem here is analogous to that in Section 3.2 and is given by:

$$\min_{K,x,y} \mathbb{E}_D\left[C(D,x(D))\right] + \sum_{r\in S} c(r)K_r, \tag{9}$$

$$s.t. \quad \mathbb{E}_D\left[g(D,x(D))\right] \geq 0, \text{ for } g \in G, \tag{10}$$

$$x_i(D) \le D_i, \text{ for } i = 1, 2, \dots, n, \tag{11}$$
$$x_i(D) \le \sum_{r \in S} y_{i,r}(D), \text{ for } i = 1, 2, \dots, n, \tag{12}$$
$$\sum_{i=1}^{n} y_{i,r}(D) \le K_r, \text{ for } r \in S, \tag{13}$$
$$y_{i,r}(D) \le K_r r_i, \text{for } i = 1, 2, \dots, n \text{ and } r \in S, \tag{14}$$

where $K = (K_r \in \mathbb{Z}_+ : r \in S)$ represents the capacity portfolio, and for each demand realization $D \in \mathbb{R}^n_+$, $y_{i,r}(D)$ denotes the capacity of resource r allocated to product type i, and $x_i(D) \in \mathbb{R}_+$ is the total satisfied demand of product type i. We consider two cases: (A) $(y_{i,r}(D)/r_i) \in \mathbb{Z}_+$, which is analogous to the setting in Section 3.2 in the sense that each unit of a resource can only be allocated to one product type; and (B) $(y_{i,r}(D)/r_i) \in \mathbb{R}_+$, in which units of a resource can be partially allocated to different product types. As before, we assume that an optimal portfolio exists.

Let $\mathcal{G}_A, \mathcal{G}_B : \mathbb{R}^n_+ \to \mathcal{P}(\mathbb{R}^n_+)$ denote the correspondences that map the resources to the demand vectors that they can satisfy for cases A and B, respectively. Then, we can write

$$\mathcal{G}_A(r) = \{y \in \mathbb{R}^n_+ | \exists p_i \in \{0, 1\} \text{ for } 1 \le i \le n \text{ such that } \sum_{i=1}^{n} p_i \le 1 \text{ and } y_i \le p_i r_i \text{ for } 1 \le i \le n\} \tag{15}$$
$$\mathcal{G}_B(r) = \{y \in \mathbb{R}^n_+ | \exists 0 \le p_i \le 1 \text{ for } 1 \le i \le n \text{ such that } \sum_{i=1}^{n} p_i \le 1 \text{ and } y_i \le p_i r_i \text{ for } 1 \le i \le n\}. \tag{16}$$

Notice that if $r_i \in \{0, 1\}$ for all i and y is restricted to the set of integers, then $\mathcal{G}_A$ reduces to the correspondence $\mathcal{F}$ defined in (8). The correspondence $\mathcal{G}_B$ generalizes the definition of $\mathcal{G}_A$ by allowing the resources to be simultaneously used to satisfy demands of different product types. We then obtain the following result.

PROPOSITION 4. *The performance regions $\mathcal{G}_A$ and $\mathcal{G}_B$ are submodular correspondences.*

As in the previous section, we next use this submodularity of the performance region to characterize properties of an optimal capacity portfolio (for both cases A and B).

PROPOSITION 5. *For a supermodular resource cost function $c(r)$, there exists an optimal capacity portfolio such that for any pair of resources u, v invested in, if $v > u$, i.e., v is either more flexible or more productive or both, then $(v - u)$ has at most one non-zero component.*

This result states that if in an optimal portfolio, a resource v dominates another resource u, and can also process more product types, then it must be the case that resource v can process exactly one more product type than resource u; furthermore, the product types that are common between the two resources must be processed by both at the same rate. This leads to two insights: first, the optimal portfolio should invest in resources that are close to each other in their skills; and further, resources that are less flexible must be more productive.

4. Conclusion

This paper generalizes the notion of submodularity for real-valued functions to correspondences. These submodular correspondences are useful in characterizing optimal flexibility portfolios in newsvendor settings. We prove that the performance regions of flexible resources in such settings are submodular correspondences. When combined with supermodular resource cost structures, this yields the result that the optimal flexibility portfolio must consist of resources with similar flexibility. For the general model in which resources may differ in their processing efficiency in addition to their flexibility, we prove that resources that are less flexible must be more efficient in the optimal portfolio.

This paper focuses on static capacity allocation decisions. However, this analysis can also be used in dynamic settings in which the resource allocation decisions in a period have no bearing on future allocations. For instance, if resources are renewed at the end of each period, or if the resources are perishable. In these scenarios, the performance region in each period is identical to that in the static settings, and hence the corresponding results carry through. Note that these results also apply to the resources in newsvendor networks (cf. Van Mieghem and Rudi, 2002).

The static performance region is also applicable when applied to multi-period problems where resources can be dynamically reassigned, as in the case of queueing networks with pre-emption. Here, the demand consists of different types of jobs or customers, each having different processing requirements and the resources do not get consumed during processing. If pre-emption is permitted and causes no losses, i.e., the resource that resumes processing on a job only needs to process the remaining work, then jobs can be transferred from one resource to another seamlessly. In this case, the performance region can be defined as the number of jobs of each type that can be in process at any time instant, and thus our results for the static setting apply. In the case that pre-emption is not practical, allowing for it can still provide a good approximation to the actual system (see the discussion in Atar (2005) for a rigorous justification in queueing systems with large volume), and thus our results may still be applicable.

Unfortunately, when resource allocation decisions made in one period may affect allocations in future periods, as in multi-period inventory models, our analysis does not carry through directly. This occurs due to the fact that the underlying performance region may now depend on the entire sample path of demand realizations and it may not be a submodular correspondence anymore. We leave this issue as a topic for future research.

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Appendix. Proofs

We provide proofs for Propositions 1, 4, and 5. Propositions 2 and 3 are special cases of Proposition 4 and 5, respectively, and we omit their proofs.

Proof of Proposition 1. Part 1 follows immediately from the definition of supermodularity.

To prove part 2, we note that if $\mathcal{F}$ is submodular, then taking the convex hull on both sides of (1), we obtain $Co(\mathcal{F}(x \vee y) + \mathcal{F}(x \wedge y)) \subseteq Co(\mathcal{F}(x)) + \mathcal{F}(y))$. So, the result follows if we prove that $Co(A) + Co(B) = Co(A+B)$ for any $A, B \in \mathcal{P}(\mathbb{R}^n)$, where $Co(X)$ denotes the convex hull of $X = A, B$. It is easy to see that $Co(A+B) \subseteq Co(A) + Co(B)$. To complete the proof, we show that $Co(A) + Co(B) \subseteq Co(A+B)$. Let $x \in Co(A) + Co(B)$ such that $x = a + b$, $a \in Co(A)$, and $b \in Co(B)$. That is, there exists m and $a_1, a_2, \cdots, a_m \in A$ and $\alpha \in \mathbb{R}^m_+$ with $\sum_{i=1}^m \alpha_i = 1$ such that $\sum_{i=1}^m \alpha_i a_i = a$ and n and $b_1, b_2, \cdots, b_n \in B$ and $\beta \in \mathbb{R}^n_+$ with $\sum_{i=1}^n \beta_i = 1$ such that $\sum_{i=1}^n \beta_i b_i = b$. We also have $a_i + b_j \in A + B$ for all $1 \le i \le m$ and $1 \le j \le n$. It then follows that $x = \sum_{i,j} \alpha_i \beta_j (a_i + b_j) \in Co(A+B)$. ■

Proof of Proposition 4. We prove that $\mathcal{G}_A$ is a submodular correspondence. Noting that $\mathcal{G}_B$ is the convex hull of $\mathcal{G}_A$, applying Proposition 1 it then follows that $\mathcal{G}_B$ is a submodular correspondence.

Denote e^i as the unit vector with its i^{th} component equal to 1, and all other components equal to zero 0. Then, noting that a resource can only be used to satisfy the demand for one product, we can write $\mathcal{G}_A(x) = \cup_{i=1}^n \{z \in \mathbb{R}^n_+ | z \le x_i e^i\}$ for $x \in \mathbb{R}^n_+$ and similarly, for $y \in \mathbb{R}^n_+$, we have $\mathcal{G}_A(y) = \cup_{i=1}^n \{z \in \mathbb{R}^n_+ | z \le y_i e^i\}$. This implies that

$$\mathcal{G}_A(x) + \mathcal{G}_A(y) = \left(\cup_{i=1}^n \{z \in \mathbb{R}^n_+ | z \le (x_i + y_i) e^i\}\right) \cup \left(\cup_{i=1}^n \cup_{j=1, j\neq i}^n \{z \in \mathbb{R}^n_+ | z \le x_i e^i + y_j e^j\}\right)$$
$$\mathcal{G}_A(x \vee y) + \mathcal{G}_A(x \wedge y) = \left(\cup_{i=1}^n \{z \in \mathbb{R}^n_+ | z \le (x_i + y_i) e^i\}\right) \cup \left(\cup_{i=1}^n \cup_{j=1, j\neq i}^n \{z \in \mathbb{R}^n_+ | z \le (x_i \vee y_i) e^i + (x_j \wedge y_j) e^j\}\right).$$

To prove that $\mathcal{G}_A(x \vee y) + \mathcal{G}_A(x \wedge y) \subseteq \mathcal{G}_A(x) + \mathcal{G}_A(y)$, it suffices to prove $\{z \in \mathbb{R}^n_+ | z \le (x_i \vee y_i) e^i + (x_j \wedge y_j) e^j\} \subseteq \{z \in \mathbb{R}^n_+ | z \le x_i e^i + y_j e^j\} \cup \{z \in \mathbb{R}^n_+ | z \le y_i e^i + x_j e^j\}$. Without loss of generality, we can assume that $x_i \vee y_i = x_i$. Then, $\{z \in \mathbb{R}^n_+ | z \le (x_i \vee y_i) e^i + (x_j \wedge y_j) e^j\} \subseteq \{z \in \mathbb{R}^n_+ | z \le x_i e^i + y_j e^j\}$ because $x_j \wedge y_j \le y_j$, and the result follows. ■

Proof of Proposition 5. Let K^* be the optimal portfolio with the smallest $\sum_{r\in S}(e'r)^2K_r$, where $e=(1,1,\cdots,1)$. We prove that K^* has the stated property arguing via contradiction. Suppose K^* invests in u and v such that $(v-u)$ has at least two non-zero components. Then, pick any j such that $v_j-u_j>0$ and define $\delta=(v_j-u_j)e^j$, where e^j is the unit vector with its j^{th} component equal to 1, and all other components equal to zero 0. Consider now a new portfolio K' constructed from K^* by replacing one unit of resource u and one unit of resource v with one unit of resource $s\equiv u+\delta$ and one unit of resource $t\equiv v-\delta=u+v-s$. Notice that $s\vee t=v$ and $s\wedge t=u$. Proposition 4 implies that K' has a performance region that is at least as large as K^*, and further because $c(r)$ is a supermodular function, the cost of the capacity portfolio K is weakly lower than that of K^*. If the cost of the capacity portfolio K is strictly lower than that of K^*, we obtain a contradiction to the optimality of K^*. Otherwise, both the capacity portfolios K^* and K have the same cost, and then noting that $e'u<e's<e'v$ and $e'u<e't<e'v$, and $e'u+e'v=e's+e't$, we find that $(e'u)^2+(e'v)^2>(e's)^2+(e't)^2$. This implies that $\sum_{r\in S}(e'r)^2K'_r<\sum_{r\in S}(e'r)^2K^*_r$, and we obtain a contradiction to the fact that K^* minimizes $\sum_{r\in S}(e'r)^2K_r$. ■